Solving Quadratic Equations

1 Quadratic Equations

Very often you will want to solve equations of the following form

$$ax^2 + bx + c = 0,$$

which means you want to find the value(s) of x that make the equation *true*. There are a number of ways to do this (and we'll see two of them here). One, solving by **factorization**, is the easiest to apply, but does not always work. The other is by using the **quadratic formula**, which yields **all** solutions (provided there are any solutions), but which involves a bit more work.

In general, you should probably *use the method of factorization when you can, and if you can't, use the quadratic formula*.

2 Solutions Using Factorization

2.1 $a = 1$

If $a = 1$ you are solving an equation of the form

$$x^2 + bx + c = 0. \tag{1}$$

To solve this, find two numbers p and q such that

1. $p + q = b$
2. $pq = c$

You can then rewrite (1) as

$$(x + p)(x + q) = 0. \tag{2}$$

So, we now have the original equation expressed as a product of two factors. The only way such a product can ever equal zero is when at least one of the two factors is zero, which means we get a solution provided

$$\begin{aligned} x + p &= 0 \qquad \textbf{or} \qquad & x + q &= 0 \\ x &= -p & x &= -q \end{aligned}$$

2.2 $a \neq 1$

If $a \neq 1$ we use nearly the same technique. We have an expression of the form

$$ax^2 + bx + c = 0, \tag{3}$$

and now we look for two numbers p and q such that

1. $p+q=b$
2. $pq=ac$ (compare this condition to the previous condition 2)

Once we have those two numbers, we have an extra step before we can get to the solution. We first rewrite (3) as

$$ax^2+px+qx+c=0. \tag{4}$$

When we're actually using this expression, you'll see that once we're at this step you can group the four terms in pairs of two and remove a common factor. This will allow you to rewrite (4) as

$$(dx+e)(fx+g)=0. \tag{5}$$

So, again we arrive at an equation expressed as a product of two things, and once again the only way such a product can ever equal zero is when at least one of the two factors is zero, which means we get a solution in this case provided

$$\begin{aligned} dx+e&=0 &\quad \textbf{or} \quad & fx+g=0 \\ x&=-\frac{e}{d} & & x=-\frac{g}{f} \end{aligned}$$

It may look a bit complicated now, but you'll see it's pretty straightforward to apply.

3 Solutions Using the Quadratic Formula

The quadratic formula always gives any solutions to a quadratic equation, but it's a bit more complicated to work with. One feature of using the quadratic equation is that there isn't any guesswork; you don't have to think of numbers with any particular special features, you just make the appropriate substitutions into the formula. So, we begin again with an equation of the form

$$ax^2+bx+c=0,$$

and the solutions are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Notice the use of the ± sign, which allows us to find *both* solutions to the equation (if indeed there are two solutions): one we get from adding the terms in the numerator, one from subtracting the terms.

The expression under the square root sign is called the **determinant**, because it's value *determines* whether or not there are two, one, or no solutions to the quadratic equation. The three cases are

1. $b^2-4ac>0$ There are two solutions.
2. $b^2-4ac=0$ There is one solution.
3. $b^2-4ac<0$ There are no solutions.

Calculating the value of the determinant could save you some work, since there's no reason the use the quadratic formula (or searching for values of p and q using the method of factorization) if you know there aren't any solutions!